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Some Multidimensional Algebras and Their Correlations

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A correlation among area-preserving diffeomorphisms, Weyl-ordered operators, vector fields, and generalized Moyal algebras in two and more dimensions is considered. A basis-independent form of the diffeomorphism algebra as well as novel infinite-dimensional algebras of the Virasoro and Floratos–Iliopoulos types are introduced.

1. INTRODUCTION

Infinite-dimensional algebras of the Virasoro [1] and Kac–Moody [1, 2] types have been of increasing interest in several branches of physics in the last few years and are under intense investigation. There are reasons to study not only one-loop, but also the multiloop or p-loop algebras. One has the two-loop algebra of Floratos and Iliopoulos [3] as an analog of the Virasoro algebra in the theory of membranes, correlations with the algebra W_∞ [4, 5], and connections to the physical states of the $c = 1$ string model [6, 7]. Integrable nonlinear equations are correlated with area-preserving diffeomorphism algebras of corresponding manifolds [8]. Some time ago the Moyal–Baker algebra [9] was proposed, which is connected to the algebra of area-preserving diffeomorphisms. The Lie algebra with trigonometric functions as structure constants [10] is also connected with the area-preserving diffeomorphism algebra. Any meaningful object existing in D-dimensional space must be invariantly defined, i.e., it must transform as a representation of Diff (D), the diffeomorphism group in D dimensions. The world-volume of a p-brane as a p-dimensional surface imbedded in D-dimensional space-

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time remains invariant under the area-preserving diffeomorphism algebra S Diff(p) after imposing all gauge conditions.

For $p = 2$ there exist at least four isomorphic algebras: the Moyal bracket algebra, the vector field algebra, the Weyl-ordered operator algebra, and the algebra of area-preserving diffeomorphisms. For $p > 2$ the correlation between these algebras is unclear; furthermore, some of them require special definitions. In this connection, in this paper two questions are considered: (1) how can the above-mentioned algebras be generalized for $p > 2$, and (2) what is their correlation?

In this work, an explicit basis-independent form of the area-preserving diffeomorphism algebra is found. The Weyl-ordered operator algebra is generalized. For $p > 2$, a new type of affine algebra is found.

2. VIRASORO-TYPE SOLUTIONS OF JACOBI IDENTITY

Consider first the general form of algebras on the p-dimensional integer lattice of indices of the Floratos–Iliopoulos type,

$$[L_m, L_n] = f(mn)L_{m+n} \tag{1}$$

where $m = (m_1, m_2, \ldots, m_p)$, $mn = A^{ij}m_i n_j$, $A^{ij} = -A^{ji}$ (antisymmetric matrix); thus $mn = -nm$ and $f(mn) = -f(-mn)$.

The Jacobi identity dictates the relation for the antisymmetric structure constants:

$$f(mn)f(mp + np) + f(np)f(nm + pm) + f(pm)f(pn + mn) = 0, \tag{2}$$

which is solved by the linear function $f(mn) = rmn + c$. The Jacobi identity admits solutions for $f(mn)$

$$\text{(a) } r\sin(kmn), \qquad \text{(b) } r\sinh(kmn), \qquad \text{(c) } r\cos(kmn) \tag{3}$$

where $r, k \in C$ are arbitrary constants, with r specified by a convenient normalization of the generators.

3. CENTRAL EXTENSION AND SUPERSYMMETRIC GENERALIZATION

The ensuing algebras which also satisfy the Jacobi identity admit central extension in the form

$$[L_m, L_n] = f(mn)\, L_{m+n} + am\delta_{m+n,0} \tag{4}$$

where a is an arbitrary p-vector.

The supersymmetric extension of algebras of type (1) is

$$[L_m, L_n] = f(mn)L_{m+n}, \qquad \{F_m, F_n\} = g(mn)L_{m+n} \tag{5}$$

and

$$[L_m, F_n] = f(mn)F_{m+n} \tag{6}$$

where F_n are fermionic generators.

For the structure constants $f(mn)$ of (3), the corresponding antisymmetric structure constants $g(mn)$ are

$$\text{(a) } s\cos(kmn), \qquad \text{(b) } s\sin(kmn), \qquad \text{(c) } s\sin(kmn) \tag{7}$$

with the condition $g(0) = 0$.

4. THE ALGEBRA OF GENERALIZED WEYL-ORDERED OPERATORS

There is a close relation between algebras with trigonometric structure constants (3) and generalized Weyl-ordered operators. Define $T_{j,m}$ as a fully symmetrized operator which can be derived from the generating function

$$\sum_{j=0}^{s} \binom{s}{j} a^j b^{s-j} T_{j,s-j} = (aP + bQ)^s, \qquad aP \equiv a^i P_i, \qquad bQ \equiv b^j Q_j \tag{8}$$

where P_i, Q_j satisfy the canonical commutation relation of the Heisenberg algebra

$$P_i Q_j - Q_j P_i = i\lambda\delta_{ij} \tag{9}$$

Then the operators

$$E_{a,b} = \frac{1}{2i\lambda} \exp \sqrt{2}i(aP + bQ) \tag{10}$$

obey the algebra

$$[E_{a,b}, E_{c,d}] = -\frac{i}{\lambda} \sinh \frac{[A, B]}{2} E_{a+c,b+d} \tag{11}$$

where $A \equiv \sqrt{2}i(aP + bQ)$, $B \equiv \sqrt{2}i(cP + dQ)$. In the case of quantum correlation of relation (11) with $(ad) \equiv a_i d^i$, $(cb) \equiv c_i b^i$ we have

$$[E_{a,b}, E_{c,d}] = \frac{i}{\lambda} \sinh[\lambda(ad) - (cb)] E_{a+c,b+d} \tag{12}$$

In the two-dimensional case this algebra turns into the Weyl-ordered two-dimensional operator algebra [11].

5. THE AREA-PRESERVING DIFFEOMORPHISM ALGEBRA

In the Σ^2 case there is a nice isomorphism between the algebra of Weyl-ordered operators and the Moyal bracket algebra:

$$[L_f, L_g] = iL_{\sin\lambda\{f,g\}} \tag{13}$$

where

$$L_f = \frac{1}{2} f(x - i\lambda\partial_y, y + i\lambda\partial_x) \tag{14}$$

We can try to find a similar relation in the multidimensional case.

In the two-dimensional case $p = 2$, to the compact surface Σ^2 with metric $h_{\alpha\beta}$ and unity area

$$\int d^2\xi \ \sqrt{|\det h_{\alpha\beta}(\xi)|} = 1 \tag{15}$$

we may introduce a complete orthonormal basis $Y_I(\xi)$ for harmonic decomposition of the surface coordinates X^μ:

$$X^\mu = \sum_I x^{\mu I} Y_I(\xi) \tag{16}$$

Then in this basis the group of area-preserving diffeomorphisms is [12]

$$[Y_A, Y_B] = f_{ABC} Y^C \tag{17}$$

where

$$f_{ABC} = \int d^2\xi \ \sqrt{|\det h_{\alpha\beta}(\xi)|} Y_A(\xi)[Y_B(\xi), Y_C(\xi)] \tag{18}$$

However, the structure constants of this representation depend on surface topology. Therefore, in the multidimensional case we derive a basis-independent area-preserving diffeomorphism algebra in terms of local differential operators.

Consider the p-dimensional surface Σ^p with local commuting coordinates x_i and $f_j \in C[\Sigma^p]$ as their differentiable functions. Then the basis-independent realization for the area-preserving diffeomorphism generators is

$$L_{\bar{f}} = \begin{vmatrix} f_{1;1} & f_{1;2} & \vdots & f_{1;p-1} & \partial_1 \\ f_{2;1} & f_{2;2} & \vdots & f_{2;p-1} & \partial_2 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ f_{p;1} & f_{p;2} & \vdots & f_{p;p-1} & \partial_p \end{vmatrix} \tag{19}$$

where $f_{i;j} \equiv \partial_j f_i(\Sigma^p)$, so that the generators $L_{\bar{f}}$ transform dx^i to $dx^i \to dx^i +$

$\partial_{(i)} a^i dx^i$ (no summing), where a^i is cofactor of ∂_i in the $L_{\bar{f}}$ expression. Infinitesimally, this is a canonical transformation which preserves the phase-space area element $dx_1 dx_2 \ldots dx_p$. The explicit form of this transformation for $p = 3$ is

$$(x_1, x_2, x_3) \rightarrow \left(x_1 + \begin{vmatrix} f_{1;2} & f_{2;2} \\ f_{1;3} & f_{2;3} \end{vmatrix}, x_2 - \begin{vmatrix} f_{1;1} & f_{2;1} \\ f_{1;3} & f_{2;3} \end{vmatrix}, x_3 + \begin{vmatrix} f_{1;1} & f_{2;1} \\ f_{1;2} & f_{2;2} \end{vmatrix} \right) \tag{20}$$

The basis-independent realization for the area-preserving diffeomorphism generators $L_{\bar{f}}$ obeys the algebra

$$[L_{\bar{f}}, L_{\bar{g}}] = L_{L_{\bar{f}}\bar{g}} - L_{L_{\bar{g}}\bar{f}} \tag{21}$$

For $p = 2$ this algebra turns into the well-known area-preserving diffeomorphisms algebra [3].

Thus, for $p > 2$ there is no isomorphism between the algebra of generalized Weyl-ordered operators and the area-preserving diffeomorphism algebra. This close relation appears only for $p = 2$. In the general case we have an algebra with two terms, and it has a more general form than our solutions of the Jacobi identity (3).

6. INTERRELATION BETWEEN THE ALGEBRAS IN THE MULTIDIMENSIONAL CASE

Thus, for $p = 2$ we have four types of algebras that are isomorphic:

1. The Moyal algebra

$$[L_f, L_g] = L_{\{f,g\}_M} \tag{22}$$

where

$$L_f = \frac{\partial f}{\partial x}\frac{\partial}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial}{\partial x}$$

and $\{f, g\}_M$ can be written as [13]

$$\{f, g\}_M = \lim_{\bar{x}' \to \bar{x}} \frac{1}{k} \sin(k\nabla \times \nabla')\, f(\bar{x}) g(\bar{x}') \tag{23}$$

$$\{f, g\}_M = \sum_{s=0}^{\infty} \frac{(-1)^s k^{2s}}{(2s+1)!} \sum_{j=0}^{2s+1} (-1)^j \binom{2s+1}{j}$$

$$\times [\partial_x^j \partial_y^{2s+1-j} f(x, y)][\partial_x^{2s+1-j} \partial_y^j g(x, y)] \tag{24}$$

$$\{f, g\}_M = \frac{1}{4\pi^2 k} \int d\bar{x}'\, d\bar{x}''\, f(\bar{x}')g(\bar{x}'')\sin k(\bar{x} \times \bar{x}' + \bar{x}' \times \bar{x}'' + \bar{x}'' \times \bar{x}) \tag{25}$$

2. The vector field algebra

$$\{L_{\bar{f}}, L_{\bar{g}}\} = L_{\{\bar{f}\bar{g}\}} \tag{26}$$

where

$$L_{\bar{f}} = \epsilon^{ij} \frac{\partial \bar{f}}{\partial x^i} \frac{\partial}{\partial x^j} \qquad \text{and} \qquad \{\bar{f}, \bar{g}\} = \epsilon^{ij} \nabla_i \bar{f} \nabla_j \bar{g}$$

3. The Weyl-ordered operator algebra

$$[E_{a,b}, E_{c,d}] = -\frac{1}{\lambda} \sin \lambda[(ad) - (cb)]E_{a+c,b+d} \tag{27}$$

4. The algebra of area-preserving diffeomorphisms

$$[L_f, L_g] = L_{\{f,g\}}; \qquad \{f, g\} \equiv (\partial f/\partial x_1)(\partial/\partial x_2) - (\partial f/\partial x_2)(\partial g/\partial x_1) \tag{28}$$

What is the interrelation among these algebras in the multidimensional case?

For $p > 2$, the forms of the algebras (26) and (27) are defined.

Let us consider an isomorphism between the possible generalizations of the Moyal and the Weyl-ordered operator algebras. For $p > 2$, let the Weyl-ordered operator algebra be defined as the algebra

$$[E_{\bar{a},\bar{b}}, E_{\bar{c},\bar{d}}] = -\frac{1}{\lambda} \sin \lambda \left[\det \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} \right] E_{\bar{a}+\bar{c},\bar{b}+\bar{d}} \tag{29}$$

where

$$\det \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = (\bar{a}\bar{d}) - (\bar{b}\bar{c})$$

Then, for $n = 2p$ the Moyal bracket can be defined as follows:

$$\{\bar{f}, \bar{g}\}_M = \lim_{\bar{x}' \to \bar{x}} r \sin[k \det(\nabla, \nabla')] \bar{f}(\bar{x})\bar{g}(\bar{x}') \tag{30}$$

which for $k \to 0$ gives $\{f, g\}_{\text{Poisson}}$.

At the same time, for any n and not only for $n = 2p$, $\{f, g\}_M$ can be defined as

$$\{\bar{f}, \bar{g}\}_M = \lim_{x' \to x} r \sin(kA^{ij}\nabla_i\nabla_j)\bar{f}(\bar{x})\bar{g}(\bar{x}') \tag{31}$$

Then, from the requirement $\lim_{k\to 0} \{\bar{f}, \bar{g}\}_M = \{\bar{f}, \bar{g}\}_{\text{Poisson}}$ we obtain that

$$\{\bar{f}, \bar{g}\}_{\text{Poisson}} = A^{ij}\nabla_i\bar{f}\nabla_j\bar{g} \tag{32}$$

From this it follows that if $A = A_{ij}dx^i \wedge dx^j$ is a closed ($dA = 0$), nonsingular ($A^p \neq 0$) 2-form, then generalizations (23) and (27) coincide. Hence, it follows that in the "noncanonical" form, for $A^{ij} \neq \epsilon^{IJ} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, the Weyl-ordered operator algebra can be written as

$$[E_{\bar{m}}, E_{\bar{n}}] = -\frac{1}{\lambda}\sin \lambda mn\, E_{\bar{m}+\bar{n}} \tag{33}$$

where $\bar{m} = \bar{a} \oplus \bar{b}$, $\bar{n} = \bar{c} \oplus \bar{d}$, $mn = A^{ij}m_i n_j$.

For $p > 2$ under what conditions is the Moyal algebra isomorphic to the algebra of area-preserving diffeomorphisms? To answer this question, let us determine the Lie algebra of vector fields on Σ^p by the condition

$$\{L_{\bar{f}}, L_{\bar{g}}\} = L_{\{\bar{f}\bar{g}\}} \tag{34}$$

where

$$L_{\bar{f}} = A^{ij}\frac{\partial \bar{f}}{\partial x^i}\frac{\partial}{\partial x^j} \qquad \text{and} \qquad \{\bar{f}, \bar{g}\} = A^{ij}\frac{\partial \bar{f}}{\partial x^i}\frac{\partial \bar{g}}{\partial x^j}$$

It is reasonable to suppose that in the case of deformation the Poisson bracket of the Lie algebra of vector fields on Σ^p will be determined by condition (31). From (34) it follows that $L_{a\{f+g,h\}} = a(L_{\{f,h\}} + L_{\{g,h\}})$ and

$$[L_{\bar{f}}, L_{\bar{g}}] = L_{L_{\bar{f}}\bar{g}} - L_{L_{\bar{g}}\bar{f}} \tag{35}$$

i.e., this is the algebra of diffeomorphisms.

To consider the opposite correlation, under what conditions is the the algebra of diffeomorphisms isomorphic to the Lie algebra of vector fields on Σ^p? Let the algebra of area-preserving diffeomorphisms be (31), where $K_{\bar{f}}$ is defined by (21). In this case, if $K_{\bar{f}}g = L_{\bar{f}}g$ and $K_{\bar{g}}f = L_{\bar{g}}f$, i.e.,

$$\det\begin{vmatrix} f_{1;1} & f_{1;2} & \vdots & f_{1;p-1} & \partial_1 \\ f_{2;1} & f_{2;2} & \vdots & f_{2;p-1} & \partial_2 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ f_{p;1} & f_{p;2} & \vdots & f_{p;p-1} & \partial_p \end{vmatrix} = \epsilon^{ij}\frac{\partial \bar{f}}{\partial x^i}\frac{\partial}{\partial x^j}, \qquad \epsilon^{ij} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{36}$$

then the algebras (37) and (31) are isomorphic.

Thus, for $p > 2$ we obtain four different algebras (25) and (29)–(31) which are isomorphic only under special additional conditions.

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